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Quality Control

- Industrial statistics deals with statistical methods most valuable in industry and with the important role they play in achieving high quality goods and services.

- Quality control is a powerful productivity technique for effective diagnosis of lack of conformity to settled standards in any of the materials, processes, machines or end products.

- Quality control, therefore, covers all the factors and processes of production which may be broadly classified as:
  - Quality of manpower
  - Quality of materials
  - Quality of machines
  - Quality of management
Controlling and improving quality has become an important business strategy for many organizations; manufacturers, distributors, transportation companies, financial services organizations; health care providers, and government agencies.

Quality is a competitive advantage. A business that can delight customers by improving and controlling quality can dominate its competitors.

This course is about the use of statistical methods and other problem-solving techniques to improve the quality of the products. These products consist of manufactured goods such as automobiles, computers, and clothing, as well as services such as the generation and distribution of electrical energy, public transportation, banking, health care, ···.
Dimensions of Quality

The quality of a product can be described and evaluated in eight different dimensions:

1. **Performance (Will the product do the intended job?)**
   - Potential customers usually evaluate a product to determine if it will perform certain specific functions and determine how well it performs them.

2. **Reliability (How often does the product fail?)**
   - Complex products, such as many appliances, automobiles, or airplanes, will usually require some repair over their service life. For example, you should expect that an automobile will require occasional repair, but if the car requires frequent repair, we say that it is unreliable.

3. **Durability (How long does the product last?)**
   - This is the effective service life of the product. Customers obviously want products that perform satisfactorily over a long period of time.
4. Serviceability (How easy is it to repair the product?)
   - There are many industries in which the customers view of quality is directly influenced by how quickly and economically a repair or routine maintenance activity can be accomplished.

5. Aesthetics (What does the product look like?)
   - This is the visual appeal of the product, often taking into account factors such as style, color, shape and other sensory features.

6. Features (What does the product do?)
   - Usually, customers associate high quality with products that have added features; that is, those that have features beyond the basic performance of the competition. For example, you might consider a spreadsheet software package to be of superior quality if it had built-in statistical analysis features while its competitors did not.
Dimensions of Quality - Contd.

7. Perceived Quality (What is the reputation of the company or its product?)

- In many cases, customers rely on the past reputation of the company concerning quality of its products. For example, if you make regular business trips using a particular airline, and the flight almost always arrives on time and the airline company does not lose or damage your luggage, you will probably prefer to fly on that carrier instead of its competitors.

8. Conformance to Standards (Is the product made exactly as the designer intended?)

- Manufactured parts that do not exactly meet the designers requirements can cause significant quality problems. An automobile consists of several thousand parts. If each one is just slightly too big or too small, many of the components will not fit together properly, and the vehicle (or its major subsystems) may not perform as the designer intended.
Definition of Quality

1. The traditional definition of quality is based on the viewpoint that products and services must meet the requirements of customers.

Definition

Quality means fitness for use.

There are two general aspects of fitness for use:

1. Quality of Design: The intensional variations in grades or levels of quality. For example automobiles differ with respect to size, appearance, and performance. These differences are the result of intentional design differences among the types of automobiles.

2. Quality of Conformance: This is how well the product conforms to the specifications required by the design. Quality of conformance is influenced by a number of factors: the choice of manufacturing processes, the training and supervision of the workforce, the types of process controls, tests, and inspection activities, ···.
2. The modern definition of quality is if variability in the important characteristics of a product decreases, the quality of the product increases.

Definition

Quality is inversely proportional to variability.

It also leads to the following definition of quality improvement:

Definition

Quality improvement is the reduction of variability in processes and products.

Excessive variability in process performance often results in waste (waste of money, time, and effort). Therefore, an alternate definition of quality improvement is the reduction of waste.
Quality Characteristics

- Every product possesses a number of elements that jointly describe what the user or consumer thinks of as quality.

- These parameters are often called quality characteristics (critical-to-quality (CTQ) characteristics).

- Quality characteristics may be of several types:
  - Physical: length, weight, voltage, viscosity/resistance
  - Sensory: taste, appearance, color
  - Time Orientation: reliability, durability, serviceability

- The different types of quality characteristics can relate directly or indirectly to the dimensions of quality.
Quality Characteristics - Contd.

- Since variability can only be described in statistical terms, statistical methods play a central role in quality improvement.

- Data on quality characteristics are typically classified as either attributes or variables.
  1. Variables data are usually continuous measurements, such as length, voltage, or viscosity/resistance.
  2. Attributes data are usually discrete data, often taking the form of counts.

- Quality characteristics are often evaluated relative to specifications.

- The specifications are the desired measurements for the quality characteristics of the components and subassemblies of the product, as well as the desired values for the quality characteristics in the final product.
The desired value for quality characteristic is called the nominal or target value.

The target value is usually bounded by a range of values that are sufficiently close to it.

- The largest allowable value for a quality characteristic is called the upper specification limit (USL).
- The smallest allowable value for a quality characteristic is called the lower specification limit (LSL).
This section is just the review of probability theory (common discrete and continuous probability distributions).

FOR MORE YOU CAN LOOK AT THE TEXT BOOK.
Three Broad Categories of SQC

1. **Statistical Process Control (SPC)**: It is a collection of problem solving tools useful in achieving process stability and improving capability through the reduction of variability. A product should be produced by a process that is stable or repeatable. More precisely, the process must be operate with little variability around the target (nominal) value of the quality characteristics. A control chart is one of the primary techniques of SPC.

2. **Design of Experiment**: A designed experiment is the approach of determining the effect controllable input factors, on the product parameters, by systematically varying these factors in the process. It is extremely helpful in discovering the key variables influencing the quality characteristics of the product. Also, statistically designed experiments are invaluable in determining the levels of the controllable variables that optimize process performance.
3. **Acceptance Sampling**: It is the inspection and classification of a sample of units selected at random from a larger batch or lot and the ultimate decision about disposition of the lot. Inspection can occur at many points in a process, usually occurs at two points: incoming raw materials or components, or final production.
Random (Chance) Causes of Variation:

- In many production processes, a certain amount of inherent or natural variability will always exist.

- This pattern results from many minor causes that behave in a random manner.

- For example, there may be slight differences in process variables like diameter, weight, service time, temperature.

- This natural variability is called a "stable system of chance causes".

- Such a variation is beyond the human control and cannot be prevented or eliminated under any circumstance.

- Chance causes of variation is tolerable and does not affect the quality and utility of the product.

- If there exist only chance causes of variation in a process, the process is said to be in statistical control.
Assignable Causes of Variation:

- The other kind of variation attributed to any production process is due to the non-random causes which affect the utility of the product.

- Such variability may arise from three sources:
  - machines (improperly adjusted, needing repair, worn tool)
  - operators (poor employee training)
  - raw materials (defective)

- Such sources of variability, which can be identified and eliminated, are called "assignable causes".

- A process that is operating in the presence of assignable causes is said to be out of control.

- The causes can be traced out from the type of defect observed in the product and the process is rectified.
Control Charts

- Statistical quality control (SQC) is a planned collection and effective use of data for studying causes of variations in quality.

- The objective of SPC is to quickly detect the occurrence of assignable causes of process shifts in order to take immediate remedial (corrective) action before many nonconforming units are manufactured.

- A control chart is a widely used technique in process control.
Control Charts - Contd.

- The center line (CL) represents the average value of the quality characteristic corresponding to the *in-control state*.

- The horizontal lines above and below from the CL are the upper control limit (UCL) and lower control limit (LCL), respectively.

- If the points are within the control limits in a *random pattern*, then the process is assumed to be *in control* and no action is required.

- However, if there is a point outside of the control limits, this indicates that the process is *out of control*.

- Thus, an investigation to find the causes responsible for this behavior is required and a corrective action is needed to eliminate the assignable causes.
Control Charts for Variables

- A single measurable quality characteristic, such as a dimension, weight, or volume, is called a variable.

- When dealing with such a variable, it is necessary to monitor both the mean value of the quality characteristic and its variability.

- The process average or mean quality level is usually controlled with the control chart for means, or the $\bar{x}$ chart.

- The process variability can be monitored with either a control chart for the range, called an $R$ chart, or a control chart for the standard deviation, called the $S$ chart.

- Therefore, it is important to maintain control over both the process mean and process variability.
Control Charts for $\bar{x}$ and $R$

- Suppose that a quality characteristic ($X$) is normally distributed with mean $\mu$ and variance $\sigma^2$, where both $\mu$ and $\sigma^2$ are known.

- If $x_1, x_2, \cdots, x_n$ is a sample of size $n$, then the sample average is

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}.$$ 

- Since $X \sim N(\mu, \sigma^2)$, then $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$.

- The probability that any sample mean falls in the control limits is $(1 - \alpha)$. That is,

$$P\left(\mu - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \leq \bar{X} \leq \mu + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha.$$
Control Charts for $\bar{x}$ and $R$ - Contd.

- It is customary to replace $z_{\alpha/2}$ by 3, so that 3-sigma limits are employed (if $z_{\alpha/2} \Rightarrow \alpha = ?$).
Control Charts for $\bar{x}$ and $R$ - Cont'd.

- Thus, the control limits for the $\bar{x}$ chart are:

  $$LCL = \mu - 3\frac{\sigma}{\sqrt{n}} = \mu - A\sigma$$

  $$CL = \mu$$

  $$UCL = \mu + 3\frac{\sigma}{\sqrt{n}} = \mu + A\sigma$$

- And the control limits for the $R$ chart are:

  $$LCL = d_2\sigma - 3d_3\sigma = (d_2 - 3d_3)\sigma = D_1\sigma$$

  $$CL = d_2\sigma$$

  $$UCL = d_2\sigma + 3d_3\sigma = (d_2 + 3d_3)\sigma = D_2\sigma$$

- The terms $A$, $D_1$ and $D_2$ are constants that depend on $n$ (are easily tabulated).
Control Charts for $\bar{x}$ and $R$ - Contd.

- In practice, $\mu$ and $\sigma^2$ are not known. Therefore, they must be estimated from preliminary samples or subgroups taken when the process is thought to be in control.

- Suppose that $m$ samples are taken, each containing $n$ observations on the quality characteristic.

- Let $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m$ be the average of each sample. Then, the grand average:

$$\bar{x} = \frac{\bar{x}_1 + \bar{x}_2 + \cdots + \bar{x}_m}{m}$$

is the best estimator of $\mu$, the process average.

- Let $R_1, R_2, \ldots, R_m$ be the ranges of the $m$ samples. The average range is

$$\bar{R} = \frac{R_1 + R_2 + \cdots + R_m}{m}.$$
Control Charts for $\bar{x}$ and $R$ - Contd.

- $m$ samples taken
- $n$ observations each sample

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Control Charts for \( \bar{x} \) and \( R \) - Contd.

- The control limits for the \( \bar{x} \) chart are:

\[
\begin{align*}
LCL &= \bar{x} - 3\hat{\sigma}_{\bar{x}} = \bar{x} - 3\frac{\hat{\sigma}}{\sqrt{n}} = \bar{x} - 3\frac{1}{\sqrt{nd_2}} R = \bar{x} - A_2 \bar{R} \\
CL &= \bar{x} \\
UCL &= \bar{x} + 3\hat{\sigma}_{\bar{x}} = \bar{x} + 3\frac{\hat{\sigma}}{\sqrt{n}} = \bar{x} + 3\frac{1}{\sqrt{nd_2}} R = \bar{x} + A_2 \bar{R}
\end{align*}
\]

- The control limits for the \( R \) chart are:

\[
\begin{align*}
LCL &= \bar{R} - 3\hat{\sigma}_R = \bar{R} - 3\frac{d_3}{d_2} \bar{R} = \left(1 - 3\frac{d_3}{d_2}\right) \bar{R} = D_3 \bar{R} \\
CL &= \bar{R} \\
UCL &= \bar{R} + 3\hat{\sigma}_R = \bar{R} + 3\frac{d_3}{d_2} \bar{R} = \left(1 + 3\frac{d_3}{d_2}\right) \bar{R} = D_4 \bar{R}
\end{align*}
\]
### Ex 1: Inside diameter of piston rings - (page 252)

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Ex 1: Solution

- It is best to begin with the R chart. Because, unless process variability is in control, the process mean limits will not have much meaning.

- Given $m = 25$ and $n = 5$. We can easily obtain $\bar{x} = 74.00118$ and $\bar{R} = 0.02324$.

- The control limits for the $\bar{x}$ chart:
  - $CL = \bar{x} = 74.00118$
  - $LCL = \bar{x} - A_2\bar{R} = 74.00118 - 0.577(0.02324) = 73.98777$
  - $UCL = \bar{x} + A_2\bar{R} = 74.00118 + 0.577(0.02324) = 74.01458$

- The control limits for the $\bar{R}$ chart:
  - $CL = \bar{R} = 0.02324$
  - $LCL = D_3\bar{R} = 0(0.02324) = 0$
  - $UCL = D_4\bar{R} = 2.114(0.02324) = 0.04912$
Ex 1: Solution - Contd.

Using Minitab
Ex 1: Solution - Contd.

- Then, the Xbar-R Chart window looks like:
Ex 1: Solution - Contd.

[Image of Xbar-R Chart interface with data entry and options]
The control charts are:

Since both the $\bar{x}$ and $R$ charts exhibit control, we would conclude that the process is in control at the stated levels and adopt the trial control limits.
Control Charts - Contd.

- Even if all the points are inside the control limits, if they behave in a systematic or nonrandom manner, then this could be an indication that the process is *out of control*.

- **Cyclic Pattern**: may result from environmental changes (temperature), operator fatigue, regular rotation of operators and/or machines, or fluctuation in voltage or other variable.
• **Shift in process level**: The shift may result from the introduction of new workers, changes in methods (raw materials, machines), or a change in either the skill, attentiveness, or motivation of the operators.
**Trend:** A trend is a continuous movement in one direction. Trends are usually due to a gradual wearing out or deterioration of a tool or some other critical process component.
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Ex 2: Solution

The control charts for the board thickness data are:
When the $R$ chart is examined it is observed that the 15th point is out of the control limits.

That is, the process variability is out of control.

We should examine this out of control point, looking for an assignable cause.

If an assignable cause is found, the point is discarded and the trail control limits are recalculated, using only the remaining points.

Suppose that an assignable cause is found for point 15. After discarding point 15, revised control limits are recalculated.
Ex 2: Solution - Contd.

[Image of Xbar-R Chart software interface]

- Observations for a subgroup are in one row of columns: C1-C3
- Data Options window: Subset
  - Include or Exclude: Specify which rows to exclude
  - Specify Which Rows To Exclude:
    - No rows
    - Rows that match: Condition...
    - Brushed rows
    - Row numbers: 15
  - Leave gaps for excluded points

(Dep’t of Stat., CCI, HU) SQC March 2015 38 / 64
Ex 2: Solution - Contd.

Xbar-R Chart of C1, ..., C3

Results exclude specified rows: 15
Now, process variability is in control when point 15 is discarded.

However, points 14 and 22 are outside the limits in the $\bar{x}$ chart.

That is, the process mean is out of control.

Suppose that assignable cause is found for both of the out of control points.

Then, we discard these points and recalculate the control limits for both $\bar{x}$ and $R$ charts.
Ex 2: Solution - Contd.
Both the revised charts indicate that process variability and mean are in control. Points 14, 15 and 22 are discarded assuming that an assignable cause is found for each point.
Control Charts for $\bar{x}$ and $S$

- Generally, $\bar{x}$ and $S$ charts are preferable to their more familiar counterparts, $\bar{x}$ and $R$ charts, when the sample size, $n$ is moderately large, say $n > 10$.

- Recall that the range method for estimating $\sigma$ loses statistical efficiency for moderate to large samples.

- $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$ is an unbiased estimator (UE) of $\sigma^2$.

- However, $S$ is not an unbiased estimator of $\sigma$.

- If underlying distribution is normal, $E(S) = c_4 \sigma$ (which is the CL for the $S$ chart when the parameters are known) and $\sqrt{V(S)} = \sigma \sqrt{1 - c_4^2}$.

- The term $c_4$ is a constant depending on the sample size, $n$. 
Control Charts for $\bar{x}$ and $S$ - Contd.

- The control charts are:

  $$LCL = c_4 \sigma - 3 \left( \sqrt{1 - c_4^2} \right) \sigma = \left( c_4 - \sqrt{1 - c_4^2} \right) \sigma = B_5 \sigma$$

  $$CL = c_4 \sigma$$

  $$UCL = c_4 \sigma + 3 \left( \sqrt{1 - c_4^2} \right) \sigma = \left( c_4 + \sqrt{1 - c_4^2} \right) \sigma = B_6 \sigma$$

- If the parameters are unknown, the control limits for $S$ chart are:

  $$LCL = \bar{S} - 3 \left( \sqrt{1 - c_4^2} \right) \frac{\bar{S}}{c_4} = \left( 1 - \left[ \sqrt{1 - c_4^2} \right] \frac{1}{c_4} \right) \bar{S} = B_3 \bar{S}$$

  $$CL = \bar{S}$$

  $$UCL = \bar{S} + 3 \left( \sqrt{1 - c_4^2} \right) \frac{\bar{S}}{c_4} = \left( 1 + \left[ \sqrt{1 - c_4^2} \right] \frac{1}{c_4} \right) \bar{S} = B_4 \bar{S}$$
Control Charts for $\bar{x}$ and $S$ - Contd.

- $E(S) = c_4 \sigma \Rightarrow E(\bar{S}) = c_4 \sigma$ where $\bar{S} = \frac{1}{m} \sum_{i=1}^{m} S_i$ where $m$ is the number of preliminary samples, each of size $n$.

- $E\left(\frac{\bar{S}}{c_4}\right) = \sigma$, thus $\frac{\bar{S}}{c_4}$ is an unbiased estimator of $\sigma$.

- Control limits on the corresponding $\bar{x}$ chart

$$LCL = \bar{x} - 3 \frac{\hat{\sigma}}{\sqrt{n}} = \bar{x} - 3 \frac{1}{c_4 \sqrt{n}} \bar{S} = \bar{x} - A_3 \bar{S}$$

$$CL = \bar{x}$$

$$UCL = \bar{x} + 3 \frac{\hat{\sigma}}{\sqrt{n}} = \bar{x} + 3 \frac{1}{c_4 \sqrt{n}} \bar{S} = \bar{x} + A_3 \bar{S}$$
Ex 3: Recall data given on Ex 1

- Can easily obtain $\bar{x} = 74.00118$ and $\bar{S} = 0.00939$.

- The control limits for the $\bar{x}$ chart are:
  
  $LCL = \bar{x} - A_3\bar{S} = 74.00118 - 1.427(0.00939) = 73.98778$
  
  $CL = \bar{x} = 74.00118$
  
  $UCL = \bar{x} + A_3\bar{S} = 74.00118 + 1.427(0.00939) = 74.01458$

- The control limits for the $S$ chart are:
  
  $LCL = B_3\bar{S} = 0(0.00939) = 0$
  
  $CL = \bar{S} = 0.00939$
  
  $UCL = B_4\bar{S} = 2.089(0.00939) = 0.01962$
Ex 3: Recall data given on Ex 1 - Contd.

- Using the $\bar{x}$ and $S$ control charts:
### Summary of the Control Charts

#### Standards given:

<table>
<thead>
<tr>
<th>Chart</th>
<th>Center Line</th>
<th>Control Limits</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{x}$ ($\mu$ and $\sigma$ known)</td>
<td>$\mu$</td>
<td>$\mu \pm A\sigma$</td>
</tr>
<tr>
<td>$R$ ($\sigma$ known)</td>
<td>$d_2\sigma$</td>
<td>$LCL = D_1\sigma$ and $UCL = D_2\sigma$</td>
</tr>
<tr>
<td>$S$ ($\sigma$ known)</td>
<td>$c_4\sigma$</td>
<td>$LCL = B_5\sigma$ and $UCL = B_6\sigma$</td>
</tr>
</tbody>
</table>

#### No standards given:

<table>
<thead>
<tr>
<th>Chart</th>
<th>Center Line</th>
<th>Control Limits</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{x}$ (using $R$)</td>
<td>$\bar{x}$</td>
<td>$\bar{x} \pm A_2R$</td>
</tr>
<tr>
<td>$\bar{x}$ (using $S$)</td>
<td>$\bar{x}$</td>
<td>$\bar{x} \pm A_3S$</td>
</tr>
<tr>
<td>$R$</td>
<td>$\bar{R}$</td>
<td>$LCL = D_3\bar{R}$ and $UCL = D_4\bar{R}$</td>
</tr>
<tr>
<td>$S$</td>
<td>$\bar{S}$</td>
<td>$LCL = B_3\bar{S}$ and $UCL = B_4\bar{S}$</td>
</tr>
</tbody>
</table>
Control Charts and Hypothesis Testing

- Suppose that the vertical axis is the sample average, $\bar{x}$. If the current value of $\bar{x}$ plots between the control limits, this indicates the process mean is in control; that is, it is equal to some value $\mu_0$. If $\bar{x}$ exceeds either control limit, the process mean is out of control; that is, it is equal to some value $\mu_1 \neq \mu_0$.

  - $H_0 : \mu = \mu_0$ (process mean is in control)
  - $H_1 : \mu \neq \mu_0$ (process mean is out of control (a shift occurs))

- The control limits of the $\bar{x}$ chart are $\mu_0 \pm L\sigma_0$; $L$ is the distance, of the control limits from the center line, in standard deviation units.

- The $CL = \mu_0$ (target or nominal value).

- If $LCL \leq \bar{x} \leq UCL$, conclude $H_0 (\mu = \mu_0)$.

- If $\bar{x} > UCL$ or $\bar{x} < LCL$, conclude $H_1 (\mu = \mu_1 \neq \mu_0)$. 
Type I Error: Concluding the process is out of control when it is really in control.

\[ \alpha = P(\text{Type I Error}) \]
\[ = P(\bar{x} > UCL \text{ or } \bar{x} < LCL / H_0) \]

Type II Error: Concluding the process is in control when it is really out of control.

\[ \beta = P(\text{Type II Error}) \]
\[ = P(LCL \leq \bar{x} \leq UCL / H_1) \]
OC Curve for an $\bar{x}$ Chart

- The ability of the $\bar{x}$ and $R$ charts to detect shifts in process quality is described by their operating characteristic (OC) curves.

- OC Curve displays probability of type II error.

- This curve gives an indication of the ability of the control chart to detect process shifts of different magnitudes.

- Suppose a quality characteristic $X$ is normally distributed with mean $\mu$ and variance $\sigma^2$. Then, $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$.

- If the mean shifts from the in-control value - say, $\mu_0$ - to another value $\mu_1 = \mu_0 + k\sigma$, the probability of not detecting this shift (not rejecting $H_0$) on the first subsequent sample or the $\beta$-risk is:

$$
\beta = P(LCL \leq \bar{X} \leq UCL \mid \mu = \mu_0 + k\sigma)
$$
OC Curve for an $\bar{x}$ Chart - Contd.

Thus,

$$
\beta = P(LCL \leq \bar{X} \leq UCL \mid \mu = \mu_0 + k\sigma)
= P\left(\frac{LCL - \mu}{\sigma/\sqrt{n}} \leq Z \leq \frac{UCL - \mu}{\sigma/\sqrt{n}}\right)
= P\left(\frac{LCL - [\mu_0 + k\sigma]}{\sigma/\sqrt{n}} \leq Z \leq \frac{UCL - [\mu_0 + k\sigma]}{\sigma/\sqrt{n}}\right)
$$

Since $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$, the control limits are $\mu_0 \pm L\frac{\sigma}{\sqrt{n}}$,

$$
\beta = P\left(\frac{[\mu_0 - L\sigma/\sqrt{n}] - [\mu_0 + k\sigma]}{\sigma/\sqrt{n}} \leq Z \leq \frac{[\mu_0 + L\sigma/\sqrt{n}] - [\mu_0 + k\sigma]}{\sigma/\sqrt{n}}\right)
= P(-L - k\sqrt{n} \leq Z \leq L - k\sqrt{n})
$$
Example 1

- Suppose we are using an $\bar{x}$ chart with the usual three-sigma limits. The sample size is $n = 5$. Determine the probability of detecting a shift to $\mu_1 = \mu_0 + 2\sigma$ on the first sample following the shift.

- We have $L = 3$, $k = 2$ and $n = 5$. Thus,

$$\beta = P(-L - k\sqrt{n} \leq Z \leq L - k\sqrt{n})$$

$$= P(-3 - 2\sqrt{5} \leq Z \leq 3 - 2\sqrt{5}) = P(-7.47 \leq Z \leq -1.47)$$

$$= P(0 \leq Z \leq 7.47) - P(0 \leq Z \leq 1.47)$$

$$= 0.5 - 0.4292 = 0.0708.$$  

- This is the $\beta$-risk, or the probability of not detecting such a shift.

- That is, the prob. of not detecting the shift on the first subsequent sample is 0.0708.
Example 1 - Contd.

- The prob. of that the shift will be detected on the first subsequent sample is \(1 - \beta = 1 - 0.0708 = 0.9292\). (This is the prob. of one point being out of control if the process is actually out of control.)

- The prob. of such a shift will be detected on the second sample is 
  \[ = \beta(1 - \beta) = 0.0658 \]

- The prob. of such a shift will be detected on the third sample is 
  \[ = \beta \beta(1 - \beta) = \beta^2(1 - \beta) = 0.0047 \]

- The prob. of such a shift will be detected on the forth sample is 
  \[ = \beta^3(1 - \beta) = 0.0003 \]

- The prob. of such a shift will be detected on the \(r^{th}\) sample is 
  \[ = \beta^{r-1}(1 - \beta) = (0.0708)^{r}(1 - 0.0708). \]
The OC curve (for the $\bar{x}$ chart) is constructed by plotting the $\beta$-risk (probability of not detecting the shift) against the magnitude of the shift ($k$) to be detected expressed in standard deviation units for different sample sizes $n$. 
The Average Run Length (ARL)

- The function $\beta^{r-1}(1 - \beta)$ is the probability distribution of a geometric random variable where the probability of success $p = 1 - \beta$.

- Recall that the geometric random variable $(Y)$ is the number of trials required until a success is occurred $R_Y = \{1, 2, \cdots\}$. For our particular case, the success is detecting the shift.

- The expected number of samples taken until the shift is detected is simply called the average run length (ARL). It is the expected value of the geometric random variable: $E(Y) = \frac{1}{1 - \beta}$.

- For the previous example, $ARL = \frac{1}{1 - 0.0708} = 1.076$.

- In other words, the ARL is the average number of points that must be plotted till a point indicates an out of control condition.
Example 2

Consider an in-control process with mean $\mu = 300$ and standard deviation $\sigma = 3$. Subgroup of size 14 are used with control limits given by: $CL = \mu$, $LCL = \mu - A\sigma$, $UCL = \mu + A\sigma$. Suppose that a shift occurs in the mean and thus the new mean is $\mu_1 = 288$. Calculate the average number of samples required (following the shift) to detect an out of control situation.

Solution: $n = 14$, $\mu = 300$ and $\sigma = 3.0$; $A = 3/\sqrt{14} = 0.8018$

- $LCL = \mu - A\sigma = 300 - 0.8018(3) = 297.59$ and
- $UCL = \mu + A\sigma = 300 + 0.8018(3) = 302.41$.

$$
\beta = P(LCL \leq \bar{X} \leq UCL \mid \mu = \mu_1 = 288)
= P(297.59 \leq \bar{X} \leq 302.41 \mid \mu = \mu_1 = 288)
= P \left( \frac{297.59 - 288}{3/\sqrt{14}} \leq Z \leq \frac{302.41 - 288}{3/\sqrt{14}} \right)
= P(11.96 \leq Z \leq 17.97) = 0
$$
Example 2 - Contd.

- The probability of not detecting the shift in the first sample is $\beta = 0$.

- Thus, the probability of detecting the shift in the first subsequent (next) sample is $1 - \beta = 1 - 0 = 1$.

- The average number of samples required to detect an out of control situation is:

$$ARL = \frac{1}{1 - \beta} = \frac{1}{1 - 0} = 1.$$  

- The shift is detected almost with certainty in the next sample.
There is no relationship between the control limits and specification limits.

The control limits are driven by the natural variability of the process (measured by the process standard deviation, $\sigma$), that is, by the natural tolerance limits of the process.

- LNTL: $3\sigma$ below the process mean
- UNTL: $3\sigma$ above the process mean
The specification limits are determined externally. These may be set by the management, the manufacturing engineers, the customer, or by product developers/designers.

Sometimes, practitioners plot specification limits on the $\bar{x}$ control chart. This practice is completely incorrect and should not be done.

When dealing with plots of individual observations (not averages), it is helpful to plot the specification limits on that chart.
Example 3

Samples of $n = 5$ units are taken from a process every hour. The $\bar{x}$ and $R$ values for a particular quality characteristics are determined as $\bar{x} = 20$ and $\bar{R} = 4.56$ after 25 samples have been collected.

- Find the 3-sigma control limits for the $\bar{x}$ and $R$ charts.
- Estimate the process standard deviation assuming that both charts exhibit control.
- Assume that the process output is normally distributed. If specifications are $19 \pm 5$, estimate the nonconforming (defects) fraction.
- If process mean shifts to 24, what is the probability of not detecting this shift on the first subsequent sample?
Example 3: Solution

- Control limits of $\bar{x}$ and $R$ charts:
  - For $\bar{x}$ Chart: $CL = \bar{x} = 20$
    - $LCL = \bar{x} - A_2 \bar{R} = 20 - 0.577(4.56) = 22.63$
    - $UCL = \bar{x} + A_2 \bar{R} = 20 + 0.577(4.56) = 22.63$
  - For $R$ Chart: $CL = \bar{R} = 4.56$
    - $LCL = D_3 \bar{R} = 0(4.56) = 0$
    - $UCL = D_4 \bar{R} = 2.115(4.56) = 9.64$

- $\hat{\sigma} = \bar{R}/d_2 = 4.56/2.326 = 1.96$

- For determining the fraction of nonconforming, we have given $LCL = 19 - 5 = 14$ and $UCL = 19 + 5 = 24$.

\[ P(x < 14 \text{ or } x > 24) = P(x < 14) + P(x > 24) \]
\[ = P\left(Z < \frac{14 - 20}{1.96}\right) + P\left(\frac{24 - 20}{1.96}\right) \]
\[ P(Z < -3.06) + P(Z > 2.05) = 0.0213 \]
Example 3 - Contd.

For the probability of not detecting the shift:

\[ \beta = P(17.37 \leq \bar{x} \leq 22.63 / \mu = 24) \]

\[ = P \left( \frac{17.37 - 24}{1.96/\sqrt{5}} \leq Z \leq \frac{22.63 - 24}{1.96/\sqrt{5}} \right) \]

\[ = P(-7.56 \leq Z \leq -1.56) \]

\[ = P(0 \leq Z \leq 7.56) - P(0 \leq Z \leq 1.56) \]

\[ = 0.5 - 0.4406 \]

\[ = 0.0594 \]
Control Charts for Attributes